

A GENERALIZED CHORDAL METRIC MAKING STRONG STABILIZABILITY A ROBUST PROPERTY

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ABSTRACT. An abstract chordal metric is defined on linear control systems described by their transfer functions. Analogous to a previous result due to Jonathan Partington [6] for H^∞ , it is shown that strong stabilizability is a robust property in this metric.

1. INTRODUCTION

The aim of this note is to give an extension of a result due to Jonathan Partington (recalled below in Proposition 1.1) saying that *strong* stabilizability is a robust property of the plant in the chordal metric. The basic and almost unique ingredient in the proof of this fact is a result proved by Partington in [5, Lemma 2.1, p.84] (which we have restated in Lemma 1.2). The only new point is that we prove that the analogous result holds in an abstract setting, hence expanding the domain of applicability from the original setting of unstable plants over H^∞ to ones over arbitrary rings of stable transfer functions satisfying mild assumptions. (Here, as is usual in the control engineering literature, H^∞ denotes the Hardy algebra of bounded holomorphic functions defined in the complex open right half plane $\{s \in \mathbb{C} : \operatorname{Re}(s) > 0\}$.)

We recall the general *stabilization problem* in control theory. Suppose that R is an integral domain with identity (thought of as the class of stable transfer functions) and let $\mathbb{F}(R)$ denote the field of fractions of R . Then the stabilization problem is:

Given $\mathbf{p} \in \mathbb{F}(R)$ (an unstable plant transfer function),
find $\mathbf{c} \in \mathbb{F}(R)$ (a stabilizing controller transfer function),
such that (the closed loop transfer function)

$$H(\mathbf{p}, \mathbf{c}) := \begin{bmatrix} \frac{\mathbf{p}}{1 - \mathbf{p}\mathbf{c}} & \frac{\mathbf{p}\mathbf{c}}{1 - \mathbf{p}\mathbf{c}} \\ \frac{\mathbf{p}\mathbf{c}}{1 - \mathbf{p}\mathbf{c}} & \frac{\mathbf{c}}{1 - \mathbf{p}\mathbf{c}} \end{bmatrix}$$

belongs to $R^{2 \times 2}$ (that is, it is stable).

The demand above that $H(\mathbf{p}, \mathbf{c}) \in R^{2 \times 2}$ guarantees that the “closed loop” transfer function of the signal map

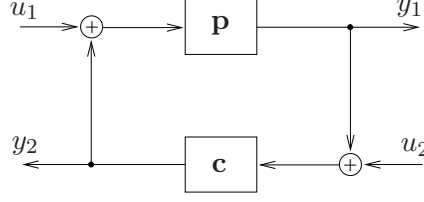
$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \mapsto \begin{bmatrix} y_1 \\ y_2 \end{bmatrix},$$

in the interconnection of \mathbf{p} and \mathbf{c} as shown in Figure 1, is stable. (So after the interconnection, “nice” signals are indeed mapped to nice signals.)

A stronger version of the problem is when we require a *stable* controller $\mathbf{c} \in R$ which stabilizes \mathbf{p} . If such a \mathbf{c} exists, then we say that \mathbf{p} is *strongly stabilizable*.

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FIGURE 1. Feedback connection of the plant \mathbf{p} with the controller \mathbf{c} .

In the *robust stabilization problem*, one goes a step further than the stabilization problem. One knows that the plant is just an approximation of reality, and so one would really like the controller \mathbf{c} to not only stabilize the *nominal* plant \mathbf{p}_0 , but also all sufficiently close plants \mathbf{p} to \mathbf{p}_0 . The question of what one means by “closeness” of plants thus arises naturally. So one needs a function d defined on pairs of stabilizable plants such that

- (1) d is a metric on the set of all stabilizable plants,
- (2) d is amenable to computation, and
- (3) stabilizability is a robust property of the plant with respect to d .

There are various known metrics which do the job, notably the gap metric ([12]), the graph metric ([10]) and the Vinnicombe ν -metric (see [11] for the rational transfer function case and [1], [9] for its recent extension for nonrational transfer functions). This last metric is in some sense the “best” one, as it is comparatively easy to compute and admits some sharp robustness results. The Vinnicombe metric itself arose from a very natural idea of defining a metric between meromorphic functions in the complex right half plane, namely the pointwise chordal metric, defined below. This metric has been studied by function theorists (see for example [4]), since it is a natural analogue of the H^∞ distance between bounded analytic functions, and it can be used for functions with poles in a disk. The use of the chordal metric to study robustness of stabilizability was made by Ahmed El-Sakkary in [8].

If $\mathbf{p}_1, \mathbf{p}_2$ are two meromorphic functions in the open right half plane, then the *chordal distance* κ between $\mathbf{p}_1, \mathbf{p}_2$ is

$$\kappa(\mathbf{p}_1, \mathbf{p}_2) := \sup_{\substack{s \in \mathbb{C}; \operatorname{Re}(s) > 0; \\ \text{either } \mathbf{p}_1(s) \neq \infty \text{ or } \mathbf{p}_2(s) \neq \infty}} \frac{|\mathbf{p}_1(s) - \mathbf{p}_2(s)|}{\sqrt{1 + |\mathbf{p}_1(s)|^2} \sqrt{1 + |\mathbf{p}_2(s)|^2}}.$$

This metric has the interpretation that it is the supremum of the pointwise Euclidean distance between the points $\mathbf{p}_1(s)$ and $\mathbf{p}_2(s)$ on the Riemann sphere. Recall that the stereographic projection allows the identification of the extended complex plane $\mathbb{C} \cup \{\infty\}$ with the unit sphere \mathbf{S} of diameter 1 in \mathbb{R}^3 , where the point $z = 0$ in the complex plane corresponds to the south pole S of the sphere \mathbf{S} and the point $z = \infty$ corresponds to the north pole N of \mathbf{S} . Points $P_{\mathbb{C}}$ in the complex plane can be identified with a corresponding point $P_{\mathbf{S}}$ on the sphere \mathbf{S} , namely the one in \mathbf{S} which lies on the straight line joining $P_{\mathbb{C}}$ and N . See Figure 1.

The following result was shown by Jonathan Partington (see [5, Theorem 2.2, p.84] or [6, Theorem 4.3.4, p.83]).

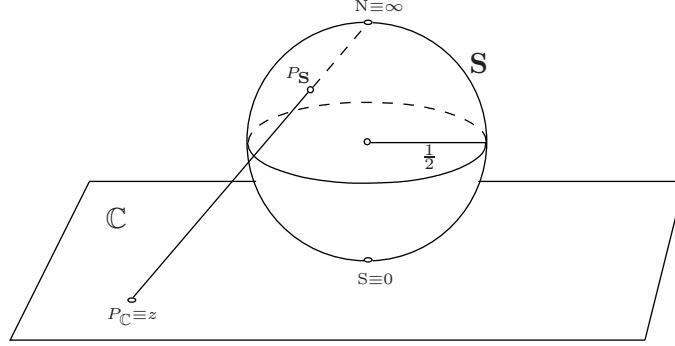


FIGURE 2. The Riemann sphere with diameter 1 and centre at $(0, 0, \frac{1}{2})$.

Proposition 1.1. *Let $\mathbf{p}_0, \mathbf{p} \in \mathbb{F}(H^\infty)$, and let $\mathbf{c} \in H^\infty$ be such that $\mathbf{g}_0 := \frac{\mathbf{p}_0}{1 - \mathbf{c}\mathbf{p}_0} \in H^\infty$. Set $k := \|\mathbf{c}\|_\infty$ and $g = \|\mathbf{g}_0\|_\infty$. If*

$$\kappa(\mathbf{p}, \mathbf{p}_0) < \frac{1}{3} \min \left\{ 1, \frac{1}{g}, \frac{1}{k(1 + kg)} \right\},$$

then \mathbf{p} is also stabilized by \mathbf{c} .

This follows from the following key estimate, which gives a lower bound on the chordal distance; see [5, Lemma 2.1, p.84] or [6, Lemma 4.3.3].

Lemma 1.2. *If $z_1, z_2 \in \mathbb{C}$ and $0 < a < 1$, then*

$$\kappa(z_1, z_2) := \frac{|z_1 - z_2|}{\sqrt{1 + |z_1|^2} \sqrt{1 + |z_2|^2}} \geq \min \left\{ \frac{a^2}{1 + a^2} |z_1 - z_2|, \frac{a^2}{1 + a^2} \left| \frac{1}{z_1} - \frac{1}{z_2} \right|, \frac{1 - a^2}{1 + a^2} \right\}.$$

1.1. Abstract set-up and main result. Our main result is given in Theorem 1.4 below. We will assume throughout the following:

- (A1) R is a commutative ring without zero divisors and with identity.
 - (A2) S is a complex, commutative, unital, semisimple Banach algebra.
 - (A3) $R \subset S$, that is, there is an injective ring homomorphism $\iota : R \rightarrow S$.
 - (A4) R is a full in S , that is, if $\mathbf{x} \in R$ and $\iota(\mathbf{x})$ is invertible in S , then \mathbf{x} is invertible in R .
- (A3) allows identification of elements of R with elements of S . So in the sequel, if \mathbf{x} is an element of R , we will simply write \mathbf{x} (an element of S !) instead of $\iota(\mathbf{x})$.

We will denote by $\mathbb{F}(R)$ the field of fractions over R . An element $\mathbf{p} \in \mathbb{F}(R)$ is said to have a *coprime factorization over R* if

$$\mathbf{p} = \frac{\mathbf{n}}{\mathbf{d}},$$

where $\mathbf{n}, \mathbf{d} \in R$, $\mathbf{d} \neq 0$ and there exist $\mathbf{x}, \mathbf{y} \in R$ such that $\mathbf{n}\mathbf{x} + \mathbf{d}\mathbf{y} = 1$.

We define the subset of *coprime factorizable plants over R* to be the set

$$\mathbb{S}(R) := \{\mathbf{p} \in \mathbb{F}(R) : \mathbf{p} \text{ has a coprime factorization}\}.$$

The maximal ideal space of S is denoted by $M(S)$. If $\mathbf{x} \in S$, then we denote by $\hat{\mathbf{x}}$ the Gelfand transform of \mathbf{x} . Also, we set

$$\|\mathbf{x}\|_\infty := \max_{\varphi \in M(S)} |\hat{\mathbf{x}}(\varphi)|.$$

If $\mathbf{p}_1, \mathbf{p}_2 \in \mathbb{S}(R)$, then the *chordal distance* κ between $\mathbf{p}_1, \mathbf{p}_2$, which have coprime factorizations

$$\mathbf{p}_1 = \frac{\mathbf{n}_1}{\mathbf{d}_1} \text{ and } \mathbf{p}_2 = \frac{\mathbf{n}_2}{\mathbf{d}_2},$$

is

$$\kappa(\mathbf{p}_1, \mathbf{p}_2) := \sup_{\varphi \in M(S)} \frac{|\widehat{\mathbf{n}}_1(\varphi)\widehat{\mathbf{d}}_2(\varphi) - \widehat{\mathbf{n}}_2(\varphi)\widehat{\mathbf{d}}_1(\varphi)|}{\sqrt{|\widehat{\mathbf{n}}_1(\varphi)|^2 + |\widehat{\mathbf{d}}_1(\varphi)|^2} \sqrt{|\widehat{\mathbf{n}}_2(\varphi)|^2 + |\widehat{\mathbf{d}}_2(\varphi)|^2}}.$$

The function κ given by the above expression is well-defined. Indeed, if

$$\mathbf{p}_1 = \frac{\mathbf{n}_1}{\mathbf{d}_1} = \frac{\widetilde{\mathbf{n}}_1}{\widetilde{\mathbf{d}}_1},$$

then $\mathbf{n}_1\widetilde{\mathbf{d}}_1 = \widetilde{\mathbf{n}}_1\mathbf{d}_1$, and so, for each $\varphi \in M(S)$, we have $\widehat{\mathbf{n}}_1(\varphi)\widehat{\widetilde{\mathbf{d}}}_1(\varphi) = \widehat{\widetilde{\mathbf{n}}}_1(\varphi)\widehat{\mathbf{d}}_1(\varphi)$. Using this one can see that

$$\frac{|\widehat{\mathbf{n}}_1(\varphi)\widehat{\mathbf{d}}_2(\varphi) - \widehat{\mathbf{n}}_2(\varphi)\widehat{\mathbf{d}}_1(\varphi)|}{\sqrt{|\widehat{\mathbf{n}}_1(\varphi)|^2 + |\widehat{\mathbf{d}}_1(\varphi)|^2}} = \frac{|\widehat{\widetilde{\mathbf{n}}}_1(\varphi)\widehat{\mathbf{d}}_2(\varphi) - \widehat{\mathbf{n}}_2(\varphi)\widehat{\widetilde{\mathbf{d}}}_1(\varphi)|}{\sqrt{|\widehat{\widetilde{\mathbf{n}}}_1(\varphi)|^2 + |\widehat{\widetilde{\mathbf{d}}}_1(\varphi)|^2}},$$

and so it follows that the expression in the definition of κ is independent of any particular choice of a coprime factorization of either plant.

We have the following result.

Proposition 1.3. κ is a metric on $\mathbb{S}(R)$.

Proof. The proof is straightforward, but we give the details as they elucidate the use of the basic assumptions in our abstract setting.

(D1) If $\mathbf{p}_1, \mathbf{p}_2 \in \mathbb{S}(R)$, then it is clear from the expression for $\kappa(\mathbf{p}_1, \mathbf{p}_2)$ that it is nonnegative. Furthermore, $\kappa(\mathbf{p}, \mathbf{p}) = 0$ for any $\mathbf{p} \in \mathbb{S}(R)$.

Finally, if $\mathbf{p}_1, \mathbf{p}_2 \in \mathbb{S}(R)$ are such that $\kappa(\mathbf{p}_1, \mathbf{p}_2) = 0$, then we must have, with $\mathbf{p}_1, \mathbf{p}_2$ having coprime factorizations

$$\mathbf{p}_1 = \frac{\mathbf{n}_1}{\mathbf{d}_1} \text{ and } \mathbf{p}_2 = \frac{\mathbf{n}_2}{\mathbf{d}_2},$$

that for all $\varphi \in M(S)$ that $\widehat{\mathbf{n}}_1(\varphi)\widehat{\mathbf{d}}_2(\varphi) - \widehat{\mathbf{n}}_2(\varphi)\widehat{\mathbf{d}}_1(\varphi) = 0$, and by (A3) and the semisimplicity of the Banach algebra (A2), we obtain $\mathbf{n}_1\mathbf{d}_2 = \mathbf{n}_2\mathbf{d}_1$, that is, $\mathbf{p}_1 = \mathbf{p}_2$.

(D2) If $\mathbf{p}_1, \mathbf{p}_2 \in \mathbb{S}(R)$, then it is clear from the expression for κ that $\kappa(\mathbf{p}_1, \mathbf{p}_2) = \kappa(\mathbf{p}_2, \mathbf{p}_1)$.

(D3) Let $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3 \in \mathbb{S}(R)$ have coprime factorizations

$$\mathbf{p}_1 = \frac{\mathbf{n}_1}{\mathbf{d}_1}, \quad \mathbf{p}_2 = \frac{\mathbf{n}_2}{\mathbf{d}_2}, \quad \mathbf{p}_3 = \frac{\mathbf{n}_3}{\mathbf{d}_3}.$$

Since the usual Euclidean distance in \mathbb{R}^3 satisfies the triangle inequality, it follows that

$$\begin{aligned} \frac{|\widehat{\mathbf{n}}_1(\varphi)\widehat{\mathbf{d}}_2(\varphi) - \widehat{\mathbf{n}}_2(\varphi)\widehat{\mathbf{d}}_1(\varphi)|}{\sqrt{|\widehat{\mathbf{n}}_1(\varphi)|^2 + |\widehat{\mathbf{d}}_1(\varphi)|^2} \sqrt{|\widehat{\mathbf{n}}_2(\varphi)|^2 + |\widehat{\mathbf{d}}_2(\varphi)|^2}} &\leq \frac{|\widehat{\mathbf{n}}_1(\varphi)\widehat{\mathbf{d}}_3(\varphi) - \widehat{\mathbf{n}}_3(\varphi)\widehat{\mathbf{d}}_1(\varphi)|}{\sqrt{|\widehat{\mathbf{n}}_1(\varphi)|^2 + |\widehat{\mathbf{d}}_1(\varphi)|^2} \sqrt{|\widehat{\mathbf{n}}_3(\varphi)|^2 + |\widehat{\mathbf{d}}_3(\varphi)|^2}} \\ &\quad + \frac{|\widehat{\mathbf{n}}_3(\varphi)\widehat{\mathbf{d}}_2(\varphi) - \widehat{\mathbf{n}}_2(\varphi)\widehat{\mathbf{d}}_3(\varphi)|}{\sqrt{|\widehat{\mathbf{n}}_3(\varphi)|^2 + |\widehat{\mathbf{d}}_3(\varphi)|^2} \sqrt{|\widehat{\mathbf{n}}_2(\varphi)|^2 + |\widehat{\mathbf{d}}_2(\varphi)|^2}} \end{aligned}$$

Consequently, $\kappa(\mathbf{p}_1, \mathbf{p}_2) \leq \kappa(\mathbf{p}_1, \mathbf{p}_3) + \kappa(\mathbf{p}_3, \mathbf{p}_2)$. This completes the proof. \square

Our main result is the following, which we will prove in the next section.

Theorem 1.4. *Suppose that $\mathbf{p}_0, \mathbf{p} \in \mathbb{S}(R)$ and $\mathbf{c} \in R$ is such that $\mathbf{g}_0 := \frac{\mathbf{p}_0}{1 - \mathbf{c}\mathbf{p}_0} \in R$. Set $k := \|\mathbf{c}\|_\infty$ and $g = \|\mathbf{g}_0\|_\infty$. If*

$$\kappa(\mathbf{p}, \mathbf{p}_0) < \frac{1}{3} \min \left\{ 1, \frac{1}{g}, \frac{1}{k(1 + kg)} \right\},$$

then \mathbf{p} is also stabilized by \mathbf{c} .

2. PROOF OF THE MAIN RESULT

Lemma 1.2 plays a key role in the proof of Theorem 1.4, and so we include its short proof (taken from [5, Lemma 2.1, p.84]) here.

Proof of Lemma 1.2. Consider the three possible cases, which are collectively exhaustive:

1° $|z_1| \leq \frac{1}{a}$ and $|z_2| \leq \frac{1}{a}$. Then $\kappa(z_1, z_2) \geq \frac{a^2}{1 + a^2} |z_1 - z_2|$.

2° $|z_1| \geq a$ and $|z_2| \geq a$. Then $\frac{1}{|z_1|} \leq \frac{1}{a}$ and $\frac{1}{|z_2|} \leq \frac{1}{a}$. As $\kappa(z_1, z_2) = \kappa\left(\frac{1}{z_1}, \frac{1}{z_2}\right)$, it follows from 1° above that $\kappa(z_1, z_2) \geq \frac{a^2}{1 + a^2} \left| \frac{1}{z_1} - \frac{1}{z_2} \right|$.

3° $|z_1| \leq a$ and $|z_2| \geq \frac{1}{a}$, or vice versa. Since the distance between the spherical caps on the Riemann sphere corresponding to the regions $\{z \in \mathbb{C} : |z| \leq a\}$ and $\left\{z \in \mathbb{C} : |z| \geq \frac{1}{a}\right\}$ is $\kappa\left(a, \frac{1}{a}\right) = \frac{1 - a^2}{1 + a^2}$, it follows that $\kappa(z_1, z_2) \geq \frac{1 - a^2}{1 + a^2}$.

This completes the proof. \square

Proof of Theorem 1.4. Let $\mathbf{p}_0 = \frac{\mathbf{n}_0}{\mathbf{d}_0}$ and $\mathbf{p} = \frac{\mathbf{n}}{\mathbf{d}}$ be coprime factorizations of \mathbf{p}_0 and \mathbf{p} .

Since \mathbf{c} stabilizes \mathbf{p}_0 , it follows in particular that

$$\frac{1}{1 - \mathbf{p}_0\mathbf{c}} = \frac{\mathbf{d}_0}{\mathbf{d}_0 - \mathbf{n}_0\mathbf{c}} \in R \text{ and } \frac{\mathbf{p}_0}{1 - \mathbf{p}_0\mathbf{c}} = \frac{\mathbf{n}_0}{\mathbf{d}_0 - \mathbf{n}_0\mathbf{c}} \in R.$$

Moreover, since $(\mathbf{n}_0, \mathbf{d}_0)$ are coprime in R , there exist $\mathbf{x}, \mathbf{y} \in R$ such that $\mathbf{n}_0 \cdot \mathbf{x} + \mathbf{d}_0 \cdot \mathbf{y} = 1$. Hence it follows that

$$\frac{1}{\mathbf{d}_0 - \mathbf{n}_0\mathbf{c}} = \frac{\mathbf{n}_0 \cdot \mathbf{x} + \mathbf{d}_0 \cdot \mathbf{y}}{\mathbf{d}_0 - \mathbf{n}_0\mathbf{c}} = \frac{\mathbf{p}_0}{1 - \mathbf{p}_0\mathbf{c}} \cdot \mathbf{x} + \frac{1}{1 - \mathbf{p}_0\mathbf{c}} \cdot \mathbf{y} \in R.$$

So $\mathbf{d}_0 - \mathbf{n}_0\mathbf{c}$ is invertible as an element of R . In particular, it is also invertible as an element of S , and so

$$\text{for all } \varphi \in M(S), \quad \widehat{\mathbf{d}_0}(\varphi) - \widehat{\mathbf{n}_0}(\varphi)\widehat{\mathbf{c}}(\varphi) \neq 0. \quad (2.1)$$

Suppose that $\mathbf{d} - \mathbf{n}\mathbf{c}$ is invertible as an element of R , then

$$\begin{aligned} \frac{1}{1 - \mathbf{p}\mathbf{c}} &= \mathbf{d} \cdot (\mathbf{d} - \mathbf{n}\mathbf{c})^{-1} \in R, & \frac{\mathbf{p}}{1 - \mathbf{p}\mathbf{c}} &= \mathbf{n} \cdot (\mathbf{d} - \mathbf{n}\mathbf{c})^{-1} \in R, \\ \frac{\mathbf{c}}{1 - \mathbf{p}\mathbf{c}} &= \mathbf{c} \cdot \mathbf{d} \cdot (\mathbf{d} - \mathbf{n}\mathbf{c})^{-1} \in R, & \frac{\mathbf{p}\mathbf{c}}{1 - \mathbf{p}\mathbf{c}} &= -1 + \mathbf{d} \cdot (\mathbf{d} - \mathbf{n}\mathbf{c})^{-1} \in R, \end{aligned}$$

and so $H(\mathbf{p}, \mathbf{c}) \in R^{2 \times 2}$, showing that \mathbf{p} is also stabilized by \mathbf{c} , and we are done.

So suppose that $\mathbf{d} - \mathbf{n}\mathbf{c}$ is not invertible as an element of R . Then $\mathbf{d} - \mathbf{n}\mathbf{c}$ is not invertible in S too, since by assumption (A4), R is a full subring of S . Thus there is a $\varphi_0 \in M(S)$ such that

$$\widehat{\mathbf{d}}(\varphi_0) - \widehat{\mathbf{n}}(\varphi_0)\widehat{\mathbf{c}}(\varphi_0) = 0. \quad (2.2)$$

We consider the following cases.

1° If $\widehat{\mathbf{d}}(\varphi_0) = 0$, then $\widehat{\mathbf{n}}(\varphi_0) \neq 0$ by the coprimeness of (\mathbf{d}, \mathbf{n}) and so by (2.2), $\widehat{\mathbf{c}}(\varphi_0) = 0$. Hence by (2.1), $\widehat{\mathbf{d}}_0(\varphi_0) \neq 0$. So in this case we have

$$\kappa(\mathbf{p}, \mathbf{p}_0) \geq \frac{|\widehat{\mathbf{d}}_0(\varphi_0)|}{\sqrt{|\widehat{\mathbf{n}}_0(\varphi_0)|^2 + |\widehat{\mathbf{d}}_0(\varphi_0)|^2}} = \kappa\left(\frac{\widehat{\mathbf{n}}_0(\varphi_0)}{\widehat{\mathbf{d}}_0(\varphi_0)}, \infty\right).$$

But since $\widehat{\mathbf{c}}(\varphi_0) = 0$, we have

$$\left|\frac{\widehat{\mathbf{n}}_0(\varphi_0)}{\widehat{\mathbf{d}}_0(\varphi_0)}\right| = \left|\frac{\widehat{\mathbf{n}}_0(\varphi_0)}{\widehat{\mathbf{d}}_0(\varphi_0) - \widehat{\mathbf{n}}_0(\varphi_0) \cdot \widehat{\mathbf{c}}(\varphi_0)}\right| = |\widehat{\mathbf{g}}_0(\varphi_0)| \leq \|\mathbf{g}_0\|_\infty = g.$$

Thus if a is any number such that $0 < a < 1$, we have

$$\kappa(\mathbf{p}, \mathbf{p}_0) \geq \kappa(g, \infty) = \kappa\left(\frac{1}{g}, 0\right) \geq \min\left\{\frac{a^2}{1+a^2}\frac{1}{g}, \frac{1-a^2}{1+a^2}\right\}. \quad (2.3)$$

2° Now let $\widehat{\mathbf{d}}(\varphi_0) \neq 0$. Then using (2.2), it follows that $\widehat{\mathbf{n}}(\varphi_0) \neq 0$ and $\widehat{\mathbf{c}}(\varphi_0) \neq 0$.

Suppose first that $\widehat{\mathbf{d}}_0(\varphi_0) = 0$. By the coprimeness of $(\mathbf{d}_0, \mathbf{n}_0)$, we have $\widehat{\mathbf{n}}_0(\varphi_0) \neq 0$. Then we have

$$\kappa(\mathbf{p}, \mathbf{p}_0) \geq \frac{|\widehat{\mathbf{d}}(\varphi_0)|}{\sqrt{|\widehat{\mathbf{n}}(\varphi_0)|^2 + |\widehat{\mathbf{d}}(\varphi_0)|^2}} = \kappa\left(\frac{\widehat{\mathbf{n}}(\varphi_0)}{\widehat{\mathbf{d}}(\varphi_0)}, \infty\right) = \kappa\left(\frac{1}{\widehat{\mathbf{c}}(\varphi_0)}, \infty\right),$$

where we have used (2.2) to obtain the last equality. But

$$g = \|\mathbf{g}_0\|_\infty = \sup_{\varphi \in M(S)} \left|\frac{\widehat{\mathbf{n}}_0(\varphi)}{\widehat{\mathbf{d}}_0(\varphi) - \widehat{\mathbf{n}}_0(\varphi)\widehat{\mathbf{c}}_0(\varphi)}\right| \geq \left|\frac{\widehat{\mathbf{n}}_0(\varphi_0)}{\widehat{\mathbf{d}}_0(\varphi_0) - \widehat{\mathbf{n}}_0(\varphi_0)\widehat{\mathbf{c}}_0(\varphi_0)}\right| = \frac{1}{|\widehat{\mathbf{c}}_0(\varphi_0)|}.$$

Thus if a is any number such that $0 < a < 1$, we have

$$\kappa(\mathbf{p}, \mathbf{p}_0) \geq \kappa\left(\frac{1}{\widehat{\mathbf{c}}(\varphi_0)}, \infty\right) \geq \kappa(g, \infty) = \kappa\left(\frac{1}{g}, 0\right) \geq \min\left\{\frac{a^2}{1+a^2}\frac{1}{g}, \frac{1-a^2}{1+a^2}\right\}. \quad (2.4)$$

Finally, suppose that $\widehat{\mathbf{d}}_0(\varphi_0) \neq 0$. If $\widehat{\mathbf{n}}_0(\varphi_0) = 0$, then

$$\kappa(\mathbf{p}, \mathbf{p}_0) \geq \frac{|\widehat{\mathbf{n}}(\varphi_0)|}{\sqrt{|\widehat{\mathbf{n}}(\varphi_0)|^2 + |\widehat{\mathbf{d}}(\varphi_0)|^2}} = \kappa\left(\frac{1}{\widehat{\mathbf{c}}(\varphi_0)}, \infty\right),$$

using (2.2), and proceeding in the same manner as above, we obtain (2.4) once again.

Suppose now that $\widehat{\mathbf{n}}_0(\varphi_0) \neq 0$. We have

$$\kappa(\mathbf{p}, \mathbf{p}_0) \geq \kappa\left(\frac{\widehat{\mathbf{n}}(\varphi_0)}{\widehat{\mathbf{d}}(\varphi_0)}, \frac{\widehat{\mathbf{n}}_0(\varphi_0)}{\widehat{\mathbf{d}}_0(\varphi_0)}\right).$$

Using (2.2) we have that

$$\frac{\widehat{\mathbf{n}}(\varphi_0)}{\widehat{\mathbf{d}}(\varphi_0)} - \frac{\widehat{\mathbf{n}}_0(\varphi_0)}{\widehat{\mathbf{d}}_0(\varphi_0)} = \frac{1}{\widehat{\mathbf{c}}(\varphi_0)} - \frac{\widehat{\mathbf{n}}_0(\varphi_0)}{\widehat{\mathbf{d}}_0(\varphi_0)} = \frac{1}{\widehat{\mathbf{c}}(\varphi_0)} \left(1 - \widehat{\mathbf{c}}(\varphi_0) \cdot \frac{\widehat{\mathbf{n}}_0(\varphi_0)}{\widehat{\mathbf{d}}_0(\varphi_0)} \right).$$

Clearly

$$\left| \frac{1}{\widehat{\mathbf{c}}(\varphi_0)} \right| \geq \frac{1}{\|\mathbf{c}\|_\infty} = \frac{1}{k}.$$

Furthermore,

$$\left| \frac{\widehat{\mathbf{d}}_0(\varphi_0)}{\widehat{\mathbf{d}}_0(\varphi_0) - \widehat{\mathbf{n}}_0(\varphi_0)\widehat{\mathbf{c}}_0(\varphi_0)} \right| = \left| 1 + \widehat{\mathbf{c}}_0(\varphi_0) \frac{\widehat{\mathbf{n}}_0(\varphi_0)}{\widehat{\mathbf{d}}_0(\varphi_0) - \widehat{\mathbf{n}}_0(\varphi_0)\widehat{\mathbf{c}}_0(\varphi_0)} \right| = |1 + \widehat{\mathbf{c}}_0(\varphi_0)\widehat{\mathbf{g}}_0(\varphi_0)| \leq 1 + kg.$$

Hence

$$\left| \frac{\widehat{\mathbf{n}}(\varphi_0)}{\widehat{\mathbf{d}}(\varphi_0)} - \frac{\widehat{\mathbf{n}}_0(\varphi_0)}{\widehat{\mathbf{d}}_0(\varphi_0)} \right| \geq \frac{1}{k(1 + kg)}. \quad (2.5)$$

Also, since $\widehat{\mathbf{n}}_0(\varphi_0) \neq 0$, we have

$$\frac{\widehat{\mathbf{d}}(\varphi_0)}{\widehat{\mathbf{n}}(\varphi_0)} - \frac{\widehat{\mathbf{d}}_0(\varphi_0)}{\widehat{\mathbf{n}}_0(\varphi_0)} = \widehat{\mathbf{c}}(\varphi_0) - \frac{\widehat{\mathbf{d}}_0(\varphi_0)}{\widehat{\mathbf{n}}_0(\varphi_0)} = -\frac{1}{\widehat{\mathbf{g}}_0(\varphi_0)}.$$

Thus

$$\left| \frac{\widehat{\mathbf{d}}(\varphi_0)}{\widehat{\mathbf{n}}(\varphi_0)} - \frac{\widehat{\mathbf{d}}_0(\varphi_0)}{\widehat{\mathbf{n}}_0(\varphi_0)} \right| \geq \frac{1}{\|\mathbf{g}_0\|_\infty} = \frac{1}{g}. \quad (2.6)$$

Combining (2.5) and (2.6), we obtain that if a is any number such that $0 < a < 1$, we have

$$\kappa(\mathbf{p}, \mathbf{p}_0) \geq \min \left\{ \frac{a^2}{1 + a^2} \frac{1}{g}, \frac{a^2}{1 + a^2} \frac{1}{k(1 + kg)}, \frac{1 - a^2}{1 + a^2} \right\}. \quad (2.7)$$

Finally, (2.3), (2.4), (2.7) yield (2.7) in *all* cases. With $a := \frac{1}{\sqrt{2}}$, we obtain

$$\kappa(\mathbf{p}, \mathbf{p}_0) \geq \frac{1}{3} \min \left\{ \frac{1}{g}, \frac{1}{k(1 + kg)}, 1 \right\},$$

which contradicts the hypothesis. Hence $\mathbf{d} - \mathbf{nc}$ is invertible as an element of R , and hence \mathbf{p} is stabilized by \mathbf{c} . \square

3. AN EXAMPLE

Consider the bidisc $\mathbb{D}^2 := \mathbb{D} \times \mathbb{D} = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < 1 \text{ and } |z_2| < 1\}$. Let $R := W^1(\mathbb{D}^2)$ be the Wiener algebra of the bidisc, that is,

$$W^1(\mathbb{D}^2) := \left\{ f := \sum_{k_1, k_2 \geq 0} a_{k_1, k_2} z_1^{k_1} z_2^{k_2} : \|f\|_1 := \sum_{k_1, k_2 \geq 0} |a_{k_1, k_2}| < +\infty \right\}.$$

Then this is a relevant class of stable transfer functions arising in the analysis/synthesis of multidimensional digital filters, and membership in this class guarantees bounded input-bounded output (BIBO) stability; see for example [2, §2.1, p.3-4].

Consider the nominal plant \mathbf{p}_0 given by

$$\mathbf{p}_0 := \frac{z_1 z_2}{z_1^2 z_2^2 - 1}$$

which has the coprime factorization $\mathbf{p} = \frac{\mathbf{n}_0}{\mathbf{d}_0}$, where $\mathbf{n}_0 := z_1 z_2$, $\mathbf{d}_0 := z_1^2 z_2^2 - 1$.

A stable controller which stabilizes \mathbf{p}_0 is $\mathbf{c} := z_1 z_2 \in W^1(\mathbb{D}^2)$, and we have

$$\mathbf{g}_0 := \frac{\mathbf{p}_0}{1 - \mathbf{p}_0 \mathbf{c}} = -z_1 z_2.$$

We take $S := A(\mathbb{D}^2)$, namely the bidisc algebra of functions continuous on $\overline{\mathbb{D}} \times \overline{\mathbb{D}}$ and holomorphic functions in \mathbb{D}^2 , with pointwise operations and the supremum norm:

$$\|f\|_\infty := \sup_{z_1, z_2 \in \mathbb{D}} |f(z_1, z_2)|, \quad f \in A(\mathbb{D}^2).$$

Then since the maximal ideal spaces of $W^1(\mathbb{D}^2)$ and of $A(\mathbb{D}^2)$ can both be identified with $\overline{\mathbb{D}} \times \overline{\mathbb{D}}$ [7, Theorem 11.7, p.279], it follows that $W^1(\mathbb{D}^2)$ is full subalgebra in $A(\mathbb{D}^2)$.

Clearly, $g := \|\mathbf{g}_0\|_\infty = \|-z_1 z_2\|_\infty = 1$ and $k := \|\mathbf{c}\|_\infty = \|z_1 z_2\|_\infty = 1$. So for all $\mathbf{p} \in \mathbb{S}(W^1(\mathbb{D}^2))$ satisfying

$$\kappa(\mathbf{p}, \mathbf{p}_0) < \frac{1}{3} \min \left\{ 1, \frac{1}{g}, \frac{1}{k(1+kg)} \right\} = \frac{1}{3} \min \left\{ 1, \frac{1}{1(1+1 \cdot 1)}, \frac{1}{1} \right\} = \frac{1}{6},$$

\mathbf{p} is also stabilized by \mathbf{c} . In particular, if we consider plants of the form

$$\mathbf{p}_\alpha := \frac{z_1 z_2 - \alpha}{z_1^2 z_2^2 - 1},$$

for real α satisfying $|\alpha| < 1$, then we can estimate $\kappa(\mathbf{p}_\alpha, \mathbf{p}_0)$ as follows. We have

$$\begin{aligned} \kappa(\mathbf{p}_\alpha, \mathbf{p}_0) &= \sup_{z_1, z_2 \in \mathbb{D}} \frac{|\alpha| |z_1^2 z_2^2 - 1|}{\sqrt{|z_1 z_1 - \alpha|^2 + |z_1^2 z_2^2 - 1|^2} \sqrt{|z_1 z_1|^2 + |z_1^2 z_2^2 - 1|^2}} \\ &\leq \sup_{z_1, z_2 \in \mathbb{D}} \frac{|\alpha|}{\sqrt{|z_1 z_1|^2 + |z_1^2 z_2^2 - 1|^2}} \leq \sup_{0 \leq k \leq 1} \frac{|\alpha|}{\sqrt{k^2 + (1 - k^2)^2}} = \frac{2}{\sqrt{3}} |\alpha|. \end{aligned}$$

Thus for α satisfying $|\alpha| < \frac{1}{4\sqrt{3}}$, \mathbf{p}_α is stabilized by \mathbf{c} .

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